Peierls instability with electron-electron interaction: the commensurate case.

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Abstract

We consider a quantum many-body model describing a system of electrons interacting with themselves and hopping from one ion to another of a one dimensional lattice. We show that the ground state energy of such system, as a functional of the ionic configurations, has local minima in correspondence of configurations described by smooth $\frac{\pi}{p_F}$ periodic functions, if the interaction is repulsive and large enough and p_F is the Fermi momentum of the electrons. This means physically that a $d = 1$ metal develop a periodic distortion of its reticular structure (Peierls instability). The minima are found solving the Eulero-Lagrange equations of the energy by a contraction method.

1 Main results

1.1 Introduction

In 1955 Peierls [P] suggested that in a one dimensional metal it would be energetically favorable to develop a periodic distortion of the linear lattice with period $\frac{p_F}{\pi}$ where p_F is the *Fermi momentum* of the conduction electrons; indeed periodic lattice distortions with period $\frac{p_F}{\pi}$ have been observed experimentally in many anisotropic compounds [F]. Theoretical attempts to understand Peierls instability are based on a *variational approach*; a functional of the ion configurations describing the total energy of the metal is introduced, and if such functional has a minimum in correspondence of $\frac{\pi}{p_F}$-periodic configuration, one says that there is Peierls instability. In [KL] the

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case $p_F = \frac{\pi}{2}$ was considered and it was indeed shown that in the *Holstein model*, describing fermions interacting with the ions of the lattice *without* electron-electron interaction, the energy has a global minimum in correspondence of a configuration of period $\frac{\pi}{p_F} = 2$. The case $p_F = \frac{\pi}{2}$ is rather special as the hamiltonian enjoys many special symmetries. In [BGM] it was treated the Holstein model with $p_F = \pi\frac{P}{Q}$, with P, Q relative prime integers, and it was shown that the Holstein model energy has local minima which correspond to $\frac{\pi}{p_F}$ periodic function, if the interaction is $\leq e|\log Q|^{-1}$ for some small ε. The Holstein model with large interaction was treated in [AAR], where Peierls instability was proved (in such regime however the minimizing periodic function has infinite many discontinuities, contrary to the small interaction case in which it is a smooth function).

There are two main open problems; the first is what happens considering $\frac{p_F}{\pi}$ as an *irrational* number (the *incommensurate* case), and the second is what is the effect of an interaction among electrons on Peierls instability. In this paper we consider the second problem; its relevance is quite clear as any realistic model must include some interaction between electrons. The only paper considering Peierls instability in presence of an electron-electron interaction is [LN], which is limited to the case $p_F = \frac{\pi}{2}$ so that one can use symmetries which are absent in the general case. We will consider the general (spinless) *commensurate* case $p_F = \pi\frac{P}{Q}$ with *weak* electron-ion and electron-electron interaction. No assumptions is done on the relative size as both the cases are interesting. There are physical situations in which the electron-ion interaction dominates over the electron-electron interaction (like in conventional low temperature superconductors) but other in which the opposite situation is found (like in high-temperature superconducting cuprates, see [FJ]).

1.2   Peierls instability

We can assume than that the units of the crystal form a liner chains with spacing $a = 1$ and the fermions, in the tight binding approximation, are hopping from one site to another. If $\Lambda = 1, 2, ..., L$ and $\psi_x^\pm$ are creation or annihilation spinless fermionic operators with periodic boundary conditions

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verifying canonical anticommutation relations the hamiltonian is,

$$H_F = \sum_{x \in \Lambda} [t_x \psi_x^+ \psi_{x+1}^- + t_x \psi_{x+1}^+ \psi_x^- - (\mu + \nu) \psi_x^+ \psi_x^-] +$$

$$+ U \sum_{x \in \Lambda} [\psi_x^+ \psi_x^- - \frac{1}{2}][\psi_{x+1}^+ \psi_{x+1}^- - \frac{1}{2}] \tag{1}$$

where μ is the chemical potential and $\mu = 1 - \cos p_F$. We assume moreover that $L = iQ$ for $i > 0$ integer (this is for preserving periodic boundary conditions, see below). The hopping matrix element t_x is a resonance integral between orbitals at site x and $x+1$ and therefore it depends on the distance ϕ_x between these sites; we can choose

$$t_x = -1 - \lambda \phi_x \tag{2}$$

Although in more refined models the lattice distortion should be treated quantum-mechanically a more common choice, which we will follow here, is to treat them in the Born-Oppenhmeier approximation by adding to the fermionic hamiltonian a term of the form $\sum_x \frac{\varphi_x^2}{2}$. The total energy of the system is then

$$F(\varphi) = \frac{1}{2} \sum_{x \in \Lambda} \varphi_x^2 + E_0(\varphi) \tag{3}$$

where $E_0(\varphi)$ is the ground state energy of the fermionic hamiltonian H_F i.e.

$$E_0(\varphi) = \lim_{\beta \to \infty} \frac{Tr H_F e^{-\beta H_F}}{Tr e^{-\beta H_F}} \tag{4}$$

We fix units so that (assuming $p_F \neq 0, \pi$) $\sin p_F = 1$. The last term in the hamiltonian takes into account the interaction between fermions with coupling U; one site interaction is forbidden by Pauli principle (the fermions are spinless) so a nearest neighbor interaction is considered. Of course $U > 0$ in order to describe Coulomb interaction, but also $U < 0$ is conceivable as a consequence of the interaction with phonons. Finally ν is a *counterterm* to be fixed to a suitable way as a function of U (see below).

We will consider $F(\varphi)$ as a functional $F : \Omega \to R$ where Ω is the class of functions defined below.

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Definition 1. Ω is the class of functions $\varphi_x : \Lambda \to R$ periodic with period $\frac{\pi}{p_F}$, even and such that

$$\varphi_x = \sum_{\substack{n=-[Q/2] \\ n \neq 0}}^{[(Q-1)/2]} \hat{\varphi}_n e^{i2p_F nx} \tag{5}$$

with $\lambda\hat{\varphi}_1 = \sigma$, $\hat{\varphi}_n = \hat{\varphi}_{-n}$ and

$$|\lambda\hat{\varphi}_n| \leq \frac{C|\sigma|}{1 + |n|^2} \tag{6}$$

with $|\sigma| < 1$. We define a norm in Ω as $||\varphi|| = \sum_{x \in \Lambda} |\varphi_x|$

The functional $F(\varphi)$ is defined over the $\frac{\pi}{p_F}$-periodic functions. As $x \in \Lambda$ then the Fourier coefficients of φ_x are $O(Q)$; if $p_F = \frac{\pi}{2}$ there is only one coefficient $\varphi_x = \hat{\varphi}_1(-1)^x$. By (6) we are requiring that φ_x has the $2p_F$ harmonic non vanishing (by such a condition if $\hat{\varphi}_1 = 0$ then φ_x is a constant). Moreover if φ is an extremal point of $F(\varphi)$, it must satisfy the condition $\hat{\varphi}_0 = \lambda\rho$ where ρ is the fermionic density. On the other hand, we can always include $\hat{\varphi}_0$ in the chemical potential μ and then we can restrict our search of local minima of $F(\varphi)$ to fields φ with zero mean. The infinite length limit is performed along a sequence of L so that periodic boundary conditions on φ_x are preserved. In the following the N depending constants (not dependent of λ, U, Q) are generically denoted by C; all the other constants are N independent. With the above notation we will prove the following theorem.

Theorem 1 *Given the Hamiltonian (1) with $p_F = \pi\frac{P}{Q}$ and P, Q positive prime, it is possible to find an ε and a function $\nu = \nu(U) = O(U)$ such that:*

a)If $U \geq 0$ (repulsive interaction) and $|\lambda|, |U| \leq \frac{\varepsilon}{|\log Q|C}$ then $F : \Omega \to R$ has a local minimum $\varphi \in \Omega$ given by (5) with

$$|\lambda\hat{\varphi}_1| = A(1 + g_2)(1 + \frac{\eta}{\lambda^2 g_1})^{-\frac{1}{\eta}}$$

$$|\lambda\hat{\varphi}_n| \leq \frac{C\lambda^{2N}}{|n|^N} \qquad n \neq 1 \tag{7}$$

where $\eta(U) = \beta_1 U + O(U^2)$, $g_2 = O(\lambda, U)$ and $g_1(U) = a_1 + O(\lambda, U)$ are continuous functions, N is a positive integer and a, β_1, C are positive constants.

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b)If $U < 0$ (attractive interaction) but $\frac{|U|}{\lambda^2 a_1} \leq \frac{1}{2}$ then, if $|\lambda|, |U| \leq \frac{\varepsilon}{|\log Q|C}$, $F : \Omega \to R$ has a local minimum $\varphi \in \Omega$ given by (5) (7).

c)If $U < 0$ (attractive interaction) but $\frac{|U|}{\lambda^2 a_1} \geq 3$ then, if $|\lambda|, |U| \leq \frac{\varepsilon}{|\log Q|C}$, there is no local minima $\varphi \in \Omega$ for $F : \Omega \to R$

In the $U = 0$ case Peierls instability is found [BGM] with $\nu = 0$, $|\lambda| \leq \frac{\varepsilon}{|\log Q|C_N}$ and $|\lambda \hat{\varphi}_1| = Ae^{-\frac{a+O(\lambda)}{\lambda^2}}$. The fact that $\nu = 0$ means that there is a simple relation between the chemical potential μ and the period of the minimizing φ_x, $i.e.\mu = 1 - \cos p_F$. If $U \neq 0$ things are different. To find a minimizing φ_x with period $\frac{\pi}{p_F}$ one fixes the chemical potential to the value $1 - \cos p_F + \nu$, where ν is a suitable counterterm. In some special case $\nu = 0$ (for instance when $p_F = \frac{\pi}{2}$) but in general ν is a non vanishing function. The above theorem says that, when U is smaller than λ the electron-electron interaction does not modify essentially the Peierls instability, as we find that the dependence of σ from λ is essentially the same as in the $U = 0$ case, as

$$|\sigma| = A[(1 + g_2)(1 + \frac{\eta}{\lambda^2 g_1})]^{-\frac{1}{\eta}} = Ae^{-\frac{a}{\lambda^2}[1+O(\frac{U}{\lambda^2})]}$$

On the other hand if U is larger than λ the electron-electron interaction has a quite dramatic effect. For attractive interaction $U < 0$ there is *no* Peierls instability (*i.e.*the electron-electron interaction destroys Peierls instability) while for repulsive interaction $U > 0$ there Peierls instability but the dependence of φ_x from λ, U is completely different with respect to the $U = 0$ case. Note also that there is a weak dependence of λ, U on Q *i.e.*$\lambda, U \leq O(\log Q^{-1})$; this means that our result can be applied to "almost incommensurate" φ_x, *i.e.*such that Q is quite large (but $\log Q$ is reasonably small). The expansion for the Ground state energy is similar to the one discussed in literature in many papers (see for istance [BM]) expect that 1)one has to understand the dependence on Q of the constants, and it is crucial to have careful bounds in order to have results valid for $\lambda, U \leq O(\log Q^{-1})$; b)one has to prove that ν is λ-independent. Finally note that the presence of the spin should not change the result when $p_F \neq \frac{\pi}{2}$ and $U > 0$.

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1.3 The contraction method

It is well known (see [BM]) that $E_0(\varphi)$ can be written as a *Grassman integral* [1]

$$E_0(\varphi) = -\lim_{\beta \to \infty} \frac{1}{L\beta} \log \int P(d\psi) e^{-UV - \lambda P - \nu N}, \tag{8}$$

where

$$V = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dx_0 \sum_{x \in \Lambda} [\psi_{\vec{x}}^+ \psi_{\vec{x}}^- - \frac{1}{2}][\psi_{\vec{x}+1}^+ \psi_{\vec{x}+1}^- - \frac{1}{2}], \tag{9}$$

$$P = -\int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dx_0 \sum_{x \in \Lambda} \varphi(x) \psi_{\vec{x}}^+ \psi_{\vec{x}}^-, \quad N = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dx_0 \sum_{x \in \Lambda} \psi_{\vec{x}}^+ \psi_{\vec{x}}^-,$$

and $\vec{x} = (x_0, x)$ and $\vec{x} + 1 = (x_0, x + 1)$. $P(d\psi)$ is a *Grassmanian integration* defined on monomials by the anticommutative Wick rule with propagator

$$g(\vec{x}; \vec{y}) = \frac{1}{\beta L} \sum_{\vec{k}} \frac{e^{-i\vec{k}(\vec{x}-\vec{y})}}{-ik_0 - \cos k + \cos p_F}, \tag{10}$$

where $\vec{k} = (k_0, k)$. (8) has a well defined $L, \beta \to \infty$ limit only if the counterterm ν is chosen in a suitable way as a function of the parameters appearing in the Hamiltonian so that the Fermi momentum is just p_F.

In order to find the minima of $F(\varphi)$ we have to differentiate with respect to $\hat{\varphi}_x$, so one has in principle to take into account the possible dependence of ν from $\hat{\varphi}_x$, which is in general very complicated. However we will show in that it is possible to choose ν as *independent* from λ and so on $\hat{\varphi}_x$. This is due to the fact that the chemical potential can be moved inside the gap opened by φ_x without affecting any physical property, and we can use this freedom to fix ν as independent of $\hat{\varphi}_x$. It follows that a necessary condition for $\varphi_x \in \Omega$ to be a local minimum for $F(\varphi)$ is that it verifies $\varphi_x = \lambda \rho_x$ where $\rho_x = \lim_{\beta \to \infty, \tau \to 0} \frac{1}{L} S^{L,\beta}(x, \tau; x, 0)$ and $S^{L,\beta}(x, \tau; x, 0)$ is the Schwinger function defined by, if $\phi_{\vec{x}}^\pm$ are Grassman variables and writing $\int d\vec{x} = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} \sum_{x \in \Lambda}$,

$$S^{L,\beta}(\vec{x}; \vec{y}) = \frac{\partial^2}{\phi_{\vec{x}}^+ \partial \phi_{\vec{y}}^-} \log \int P(d\psi) e^{-\mathcal{V}(\psi) - \int d\vec{x}[\phi_{\vec{x}}^+ \psi_{\vec{x}}^- + \phi_{\vec{x}}^- \psi_{\vec{x}}^+]}|_{\phi=0}, \tag{11}$$

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[1] We use the same symbol ψ for field and Grassman variables with a traditional abuse of notation.

and $\mathcal{V} = UV + \lambda P + \nu N$. Note that when L is finite $F(\varphi)$ can be considered as an L-dimensional function, as φ_x is determined by the values of $\varphi_1, \varphi_2, ... \varphi_L$. We will see that $F : \Omega \to R$ is differentiable so that the minima of $F : \Omega \to R$ verify, for $n \neq 0$, $n = -[Q/2], ..., [(Q-1)/2]$, verify

$$\hat{\varphi}_n = \lambda \hat{\rho}_n(\varphi) \tag{12}$$

and $M_{n,m}$ positive definite, where

$$M_{n,m} = \delta_{n,m} - \lambda \frac{\partial}{\partial \hat{\varphi}_m} \hat{\rho}_n(\varphi)$$

and $\rho(x) = \sum_{\substack{n=-[Q/2] \\ n \neq 0}}^{[(Q-1)/2]} \rho(x) e^{i 2 p_F n x}$ where

$$\rho(x) = \lim_{\beta \to \infty} \frac{Tr \psi_x^+ \psi_x^- e^{-\beta H_F}}{Tr e^{-\beta H_F}} \tag{13}$$

(12) are the *Eulero-Lagrange* equations of our variational problem.

We will see in §2.3,§2.4 that (12) can be written as

$$\hat{\varphi}_n = -\lambda^2 c_n(\sigma) \hat{\varphi}_n + \lambda \tilde{\rho}_n(\sigma, \Phi) , \quad \sigma = \lambda \hat{\varphi}_1 , \quad \Phi = \{\lambda \hat{\varphi}_n\}_{|n|>1} \tag{14}$$

where $c_n(\sigma)$ depends on φ only through σ.

The equation (14) has of course the trivial solution $\hat{\varphi}_n = 0, \forall n$, but it is easy to see that this is not a local minimum. Therefore we shall look for solutions such that $\sigma \neq 0$, so that we can rewrite (14) as

$$(1 + \lambda^2 c_1(\sigma)) = \frac{\lambda^2 \tilde{\rho}_1(\sigma, \Phi)}{\sigma}$$

$$\Phi_n = \lambda \hat{\varphi}_n = \frac{\lambda^2 \tilde{\rho}_n(\sigma, \Phi)}{(1 + \lambda^2 c_n(\sigma))} , \qquad |n| > 1 \tag{15}$$

Explicit bounds for the quantities appearing in the above equation are given in the following lemma.

Lemma 1 *If $\varphi_x \in \Omega$ there exists a (Q-independent) ε and a $\eta(U), \nu(U)$ independent from λ such that, $|\sigma| \leq \frac{\varepsilon^2}{Q^4}$ and $|U| \leq \varepsilon$ then*

$$|\tilde{\rho}_n| \leq \frac{C|\sigma|}{|n|^N} \quad |n| \neq 0$$

$$|c_n(\sigma)| \leq C|\log Q|Q^{C|U|} \quad |n| \neq 1 \qquad (16)$$

$$c_1(\sigma) = \frac{1}{\eta}\left[\left(\frac{|\sigma|}{A}\right)^{-\eta} - 1\right][a_1 + UF] + \sigma f$$

and $|F|, |\tilde{\rho}_1|, |f| \leq C$ and $\eta = \beta U + O(U^2)$

We find a solution of (15) considering σ as a variable. We solve the second of (15) by considering σ as a parameter and assuming $\lambda^2, |U| \leq \frac{\varepsilon}{C_N logQ}$. Fixed $L = L_i$, Φ is a finite sequence of $Q - 3$ elements, which can be thought as a vector in R^{Q-3}, which is a function of σ. We consider the space $\mathcal{F} = C^1(R^{Q-3})$ of C^1-functions of σ with values in R^{Q-3}; the solutions of eq(15) can be seen as fixed points of the operator $\mathbf{T}_\lambda : \mathcal{F} \to \mathcal{F}$, defined by the equation:

$$[\mathbf{T}_\lambda(\Phi)]_n(\sigma) = \frac{\lambda^2 \tilde{\rho}_n(\sigma, \Phi(\sigma))}{(1 + \lambda^2 c_n(\sigma))}$$

We shall define, for each positive integer N, a norm in $\mathcal{F}$ in the following way:

$$||\Phi||_{\mathcal{F}} = sup_{|n|>1}\left\{|n|^N\left[|\sigma|^{-1}|\Phi_n(\sigma)| + \left|\frac{\partial \Phi_n}{\partial \sigma}(\sigma)\right|\right]\right\}$$

We shall also define

$$\mathcal{B} = \{\Phi \in \mathcal{F} : ||\Phi||_{\mathcal{F}} \leq 1\}$$

$$R(\Phi)_n(\sigma) = \tilde{\rho}_n(\sigma, \Phi(\sigma)) , \quad |n| \geq 2$$

It is an easy corollary of lemma 1 that, if if $\Phi, \Phi' \in \mathcal{B}$ and then

$$||R(\Phi) - R(\Phi')||_{\mathcal{F}} \leq C_N||\Phi - \Phi'||_{\mathcal{F}}$$

and

$$||R(0)||_{\mathcal{F}} \leq C \sup_{|n|>1}\left\{|n|^N|\sigma|^{\frac{|n|}{10}}\right\}$$

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It follows that

Lemma 2. *There are ε, c, K, independent of N, such that for $\lambda, |U| \le \frac{\varepsilon}{C \log Q}$ there exists a unique solution $\Phi \in \mathcal{B}$ of the second of (15); moreover the solution satisfies the bound*

$$\|\Phi\|_{\mathcal{F}} \le \lambda^{2N}$$

Proof of Lemma. If $\lambda^2, |U| \le \frac{\varepsilon}{C \log Q}$ then from (16)

$$\lambda^2 |c_n(\sigma)| \le C \lambda^2 |\log Q| Q^{C|U|} \le 2\varepsilon$$

so that for ε small enough

$$\frac{\lambda^2}{1 + \lambda^2 c_n(\sigma)} \le 2\lambda^2$$

so that

$$\|\mathbf{T}_\lambda(\Phi)\|_{\mathcal{F}} \le C_N \lambda^2 \le 1$$

which means that $\mathcal{B}$ is invariant under the action of $\mathbf{T}_\lambda$. Moreover if $\Phi, \Phi' \in \mathcal{B}$ for λ small enough

$$\|\mathbf{T}_\lambda(\Phi) - \mathbf{T}_\lambda(\Phi')\|_{\mathcal{F}} \le C\lambda^2 |\Phi - \Phi'|_{\mathcal{F}} \le \frac{1}{2}|\Phi - \Phi'|_{\mathcal{F}}$$

so that T is a contraction on $\mathcal{B}$. Hence, by the contraction mapping principle, there is a unique fixed point $\bar{\Phi}$ of $\mathbf{T}_\lambda$ in $\mathcal{B}$, which can be obtained as the limit of the sequence $\Phi^{(k)}$ defined through the recurrence equation $\Phi^{(k+1)} = \mathbf{T}_\lambda(\Phi^{(k)})$, with any initial condition $\Phi^{(0)} \in \mathcal{B}$. If we choose $\Phi^{(0)} = 0$, we get

$$|\bar{\Phi}|_{\mathcal{F}} \le \sum_{i=1}^{\infty} \|\Phi^{(i)} - \Phi^{(i-1)}\|_{\mathcal{F}} \le \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} \|\Phi^{(1)}\|_{\mathcal{F}} \le \|\Phi^{(1)}\|_{\mathcal{F}}$$

On the other hand

$$\|\Phi^{(1)}\|_{\mathcal{F}} = \|\mathbf{T}_\lambda(0)\|_{\mathcal{F}} \le C_N \left(\lambda^2\right)^N$$

which immediately implies that there is a solution belonging to Ω.

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We insert in the first of (15) $\hat{\varphi}_n$ as a function of σ and we call simply

$$\tilde{\rho}_1(\sigma, \hat{\varphi}_n(\sigma)) \equiv \rho_1(\sigma)$$

and we look for a solution σ.

Lemma 3 *Assume that* $|U|, \lambda^2 \leq \frac{\varepsilon}{C \log Q}$.
1)If $U > 0$, *or if* $U < 0$ *but* $\frac{|U|}{\lambda^2 a_1} < \frac{1}{2}$ *then there exists a solution of* $\sigma = \lambda^2 \rho_1(\sigma)$ *such that* $|\sigma|Q^4 \leq \varepsilon^2$
2)If $\frac{|U|}{\lambda^2 a_1} > 3$ *but* $U < 0$ *there is no solution such that* $|\sigma| \leq 1$.

Proof of lemma 3 We write the self-consistence equation as

$$\frac{\eta}{\lambda^2} = [\left(\frac{A}{|\sigma|}\right)^{\eta} - 1][a^{-1} + UF] + \sigma\tilde{f}$$

where η, F are λ-independent and $|F|, |\tilde{f}| \leq C$. We look for a solution of the form
$$|\sigma| = A[(1 + g_2)(1 + \frac{\eta}{\lambda^2(a_1 + UF)})]^{-\frac{1}{\eta}}$$

with $g_2 = g_2(\lambda, U)$ and $O(\varepsilon)$. Substituing by the implicit function theorem there is a solution for $g_2 = O(\varepsilon)$ if $(1 + \frac{\eta}{\lambda^2(a^{-1}+Uf_1)}) \leq \frac{1}{2}$; this is true assuming that $U \geq 0$ or $U < 0$ and $\frac{a|\eta|}{\lambda^2} \leq \frac{1}{2}$ or $\frac{a|\eta|}{\lambda^2} \geq 3$.

Finally we check that $|\sigma|Q^4 \leq \varepsilon^2$ if $|U|, \lambda^2 \leq \frac{\varepsilon}{\log Q}$. Note in fact that if $U > 0, U \leq \frac{\varepsilon}{\log Q}$ then, for ε small enough

$$|\sigma| = A(1 + \frac{\eta}{\lambda^2})^{-\frac{1}{\eta}} = Ae^{-\frac{1}{\lambda^2}[\frac{\lambda^2}{U}\log(1+\frac{\eta}{\lambda^2})]} \leq Ae^{-\frac{1}{\lambda^2}} \leq \frac{1}{Q^{\frac{1}{\varepsilon}}} \leq \frac{\varepsilon^2}{Q^4}$$

The same bound is true if $U < 0, \frac{\eta}{\lambda^2} \leq \frac{1}{2}$.
On the other hand if $|\frac{U}{\lambda^2}| > 3, U < 0, U \leq \frac{\varepsilon}{\log Q}$ surely $|\sigma| \geq 2^{\frac{1}{\varepsilon}} > 2$ for ε small enough.

Theorem 1 is an easy consequence of the above lemmas.

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2 Renormalization Group analysis

2.1 The effective potential

We start by studying the *free energy*

$$E_{L,\beta} = -\frac{1}{L\beta} \log \int P(d\psi^{[h,0]}) e^{-V(\psi^{[h,0]})} \tag{17}$$

with $\lim_{\beta \to \infty} E_{L,\beta}$ giving the ground state energy. The integration can be performed iteratively by a slight modification of the procedure described in [BM]. It is convenient to decompose the Grassman integration $P(d\psi)$ into a product of independent integrations. Let be $|\vec{k}| = \sqrt{k_0^2 + ||k||_T^2}$. We write

$$g(\vec{k}) = f_1(\vec{k})g(\vec{k}) + (1 - f_1(\vec{k}))g(\vec{k}) =$$
$$g^{(u.v.)}(\vec{k}) + g^{(i.r.)}(\vec{k}) \tag{18}$$

where $f_1(\vec{k}) = 1 - \chi(k - p_F, k_0) - \chi(k + p_F, k_0)$ and $\chi(k', k)$ is a C^∞ function with compact support such that it is 1 for $|\vec{k'}| \le \frac{a_0}{\gamma}$ and 0 for $|\vec{k'}| > a_0$, where $\gamma > 1$ and γ, a_0 are chosen so that $\chi(k \pm p_F, k_0)$ are non vanishing only in two non overlapping regions around $\pm p_F$. We write $k = k' + \omega p_F$, $\omega = \pm 1$ and

$$g^{(i.r.)}(\vec{k}) = \sum_{\omega = \pm 1} \sum_{h=-\infty}^{0} f_h(\vec{k'})g(\vec{k}) \equiv \sum_{\omega = \pm 1} \sum_{h=-\infty}^{0} g^{(h)}(\vec{k})$$

where $f_h(\vec{k'}) = \chi(\gamma^{-h}\vec{k'}) - \chi(\gamma^{-h+1}\vec{k'})$ has support $O(\gamma^h)$. Then we can see $P(d\psi)$ as the product of independent integrations of fields

$$P(d\psi) = P(d\psi^{(1)}) \prod_{h=-\infty}^{0} \prod_{\omega = \pm 1} P(d\psi_\omega^{(h)})$$

and $\psi_{\vec{k}}^\sigma = \psi_{\vec{k}}^{\sigma(1)} + \sum_{h=-\infty}^{0} \sum_{\omega = \pm 1} \psi_{\vec{k},\omega}^{\sigma(h)}$. The field $\psi_{\vec{k},\omega}^{\sigma(h)}$ has momenta distant $O(\gamma^h)$ to ωp_F. The idea is to integrate iteratively $\psi^{(1)}, \psi^{(0)}, \psi^{(-1)}.....$ going closer and closer to the Fermi surface. The integration of $\psi^{(1)}$, the *ultraviolet* integration, gives

$$e^{-\mathcal{V}^{(0)}(\psi^{(\le 0)})} = \int P(d\psi^{(1)}) e^{-\mathcal{V}^{(0)}(\psi^{(1)} + \psi^{(\le 0)})}$$

where (for shortening the notation $\frac{1}{\beta L}\sum_{\vec{k}}$ will be denoted by $\int d\vec{k}$)

$$\mathcal{V}^{(0)}(\psi^{(\leq 0)}) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\int d\vec{k}_1...d\vec{k}_{2n}\psi_{\vec{k}_1}^{(\leq 0)\sigma_1}...\psi_{\vec{k}_{2n}}^{(\leq 0)\sigma_n}$$

$$W_{2n,m}^0(\vec{k}_1,...,\vec{k}_{2n})\delta(\sum_{i=1}^{2n}\sigma_i\vec{k}_i + 2m\vec{p}_F) \tag{19}$$

where if $\vec{p}_F = (p_F, 0)$, $\sigma_i = \pm$ and the kernels $W_{n,m}^0(\vec{k}_1,...,\vec{k}_n; z)$ are C^{∞} bounded functions such that $W_{n,m}^0 = W_{n,-m}^0$ and $|W_{n,m}^0| \leq C^n z^{\max(2,n/2-1)}$ if $z = Max(|\lambda|, |U|, |\nu|)$. By an explicit computation it follows that $W_{4,0}^0 = U + O(U^2)$, $W_{4,m}^0 = O(U\sigma)$ for $m \neq 0$ and $W_{2,m}^0 = \sigma + O(\sigma U)$ for $m \neq 0$. $\mathcal{V}^{(0)}$ is called *effective potential at scale* 0; note that it contains non local interactions between an arbitrary number of fermions.

The study of the *infrared* integration is much more involved. The integration is performed iteratively, setting $Z_0 = 1$, $\sigma_0 = \sigma$, in the following way: once that the fields $\psi^{(0)},...,\psi^{(h+1)}$ have been integrated we have

$$\int P_{Z_h}(d\psi^{(\leq h)})\, e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})} \tag{20}$$

with $C_h(\vec{k}')^{-1} = \sum_{j=-\infty}^{h} f_j(\vec{k}')$ and $\alpha(k') = (\cos k' - 1)\cos p_F$, $v_0 = \sin p_F$:

$$P_{Z_h}(d\psi^{(\leq h)}) = \prod_{\vec{k}'}\prod_{\omega=\pm 1} d\psi_{\vec{k}'+\omega\vec{p}_F,\omega}^{(\leq h)+} d\psi_{\vec{k}'+\omega\vec{p}_F,\omega}^{(\leq h)-} \tag{21}$$

$$\exp\left\{ -\sum_{\omega=\pm 1}\int d\vec{k}' C_h(\vec{k}')Z_h\Big[\big(-ik_0 - \alpha(k') + \omega v_0\sin k'\big)\right.$$

$$\left.\psi_{\vec{k}'+\omega\vec{p}_F,\omega}^{(\leq 0)+}\psi_{\vec{k}'+\omega\vec{p}_F,\omega}^{(\leq 0)-} - \sigma_h(\vec{k}')\,\psi_{\vec{k}'+\omega\vec{p}_F,\omega}^{(\leq 0)+}\,\psi_{\vec{k}'-\omega\vec{p}_F,-\omega}^{(\leq 0)-}\Big]\right\}$$

Note that after h steps the integration is different with respect to the initial one; there is a wave function renormalization Z_h and a mass term σ_h. Moreover the effective potential at scale h has the form

$$\mathcal{V}^{(h)}(\psi^{(\leq h)}) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int d\vec{k}_1...d\vec{k}_{2n}\prod_{i=1}^{n}\psi_{\vec{k}_i'+\omega_i\vec{p}_F,\omega_i}^{\sigma_i(\leq h)} \tag{22}$$

$$d(\sum_{i=1}^{2n}\sigma_i(\vec{k}_i' + \omega_i\vec{p}_F) + 2m\vec{p}_F)W_{n,m}^h(\vec{k}_1' + \vec{\omega}_1\vec{p}_F, ..; , ..; , ..; , ..; , ..; , ..; , ..; \{\omega\})$$

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In order to integrate $\psi^{(h)}$ we write $\mathcal{V}^{(h)}$ as $\mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$, with $\mathcal{R} = 1 - \mathcal{L}$. The $\mathcal{L}$ operation is defined to extract the non irrelevant terms in $\mathcal{V}^{(h)}$; it is easy to check from a power counting argument that the terms in $\mathcal{V}^{(h)}$ involving six or more fields are irrelevant then $\mathcal{L} = 0$ on such terms. Moreover we will define $\mathcal{L} = 0$ on the addenda in eq(22) *not* verifying the condition

$$\sum_{i=1}^{2n} \sigma_i \omega_i p_F + 2mp_F = 0 \quad \text{mod. } 2\pi \tag{23}$$

which means that we are considering irrelevant the terms such that the sum of momenta measured from the Fermi surface is not vanishing (but it can be arbitrary small, due to the irrationality of $\frac{p_F}{\pi}$). In conclusion the definition of $\mathcal{L}$ is the following

1) If $2n > 4$ then
$$\mathcal{L}W_{2n,m}^h(\vec{k}_1, ...) = 0$$

2) If $2n = 4$ then
$$\mathcal{L}W_{4,m}^h(\vec{k}_1, ...) = \delta_{m,0}\delta_{\sum_{i=1}^{4}\sigma_i\vec{\omega}_i,0}W_{4,m}^h(\omega_1\vec{p}_F, ..., \omega_4\vec{p}_F) \tag{24}$$

3) If $2n = 2$, $\omega_1 = \omega_2$ then
$$\mathcal{L}\{W_{2,m}^h(\vec{k}_1' + \omega_1\vec{p}_F, \vec{k}_2' + \omega_2\vec{p}_F) = \delta_{m,0}[W_{2,m}^h(\omega_1\vec{p}_F, \omega_2\vec{p}_F)$$
$$+\omega_1 E(k' + \omega_1 p_F)\partial_k W_{2,m}^h(\omega_1\vec{p}_F, \omega_2\vec{p}_F) +$$
$$k^0\partial_{k_0}W_{2,m}^h(\omega_1\vec{p}_F, \omega_2\vec{p}_F)] \tag{25}$$

where $E(k' + \omega p_F) = v_0\omega \sin k' + (1 - \cos k')\cos p_F$ and the symbol $\partial_k, \partial_{k_0}$ means discrete derivatives.

4) If $n = 2$, $\omega_1 = -\omega_2$ then
$$\mathcal{L}W_{2,m}^h(\vec{k}_1' + \omega_1\vec{p}_F, \vec{k}_2' + \omega_2\vec{p}_F) = \delta_{m,\omega_2}W_{2,m}^h(\omega_1\vec{p}_F, \omega_2\vec{p}_F) \tag{26}$$

The Kronecker deltas in the r.h.s. of (24), (25)(26) ensure that $\mathcal{L} = 0$ if (23) is not verified. We find

$$\mathcal{L}\mathcal{V}^{(h)}(\psi) = \gamma^h n_h F_\nu^{(\leq h)} + s_h F_\sigma^{(\leq h)} + z_h F_\zeta^{(\leq h)} + a_h F_\alpha^{(\leq h)} + u_h F_U^{(\leq h)}$$

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where

$$F_\sigma^{(\leq h)} = \sum_{\omega=\pm 1} \int d\vec{k}' \psi_{\vec{k}'+\omega\vec{p}_F,\omega}^{(\leq h)+} \psi_{\vec{k}'-\omega\vec{p}_F,-\omega}^{(\leq h)-}$$

$$F_i^{(\leq h)} = \sum_{\omega=\pm 1} \int d\vec{k}' f_i(\vec{k}') \psi_{\vec{k}'+\omega\vec{p}_F,\omega}^{(\leq h)+} \psi_{\vec{k}'+\omega\vec{p}_F,\omega}^{(\leq h)-}$$

$$F_U^{(\leq h)} = \int [\prod_{i=1}^4 d\vec{k}_i'] \delta(\sum_{i=1}^4 \sigma_i \vec{k}_i) \times$$

$$\psi_{\vec{k}_1'+\vec{p}_F,1}^{(\leq h)+} \psi_{\vec{k}_2'+\vec{p}_F,1}^{(\leq h)-} \psi_{\vec{k}_3'-\vec{p}_F,-1}^{(\leq 0)+} \psi_{\vec{k}_4'-\vec{p}_F,-1}^{(\leq 0)-} \delta(\sum_{i=1}^4 \sigma_i \vec{k}_i)$$

where $i = \nu, \zeta, \alpha$ and $f_\nu = 1$, $f_\zeta = -ik_0$ and $f_\alpha = E(k' + \omega p_F)$; moreover $u_0 = U(\hat{v}(0) - \hat{v}(2p_F)) + O(U^2)$, $s_0 = O(U\lambda)$, $a_0, z_0 = O(U^2)$, $n_0 = \nu + O(U)$. Note that in $\mathcal{L}V^{(h)}$ there are terms renormalizing mass and the wave function renormalization and it is convenient to include them in the fermionic integration writing

$$\int P_{Z_h}(d\psi^{(\leq h)}) e^{-\mathcal{V}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})} = \int \tilde{P}_{Z_{h-1}}(d\psi^{(\leq h)}) e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})} \qquad (27)$$

where $\tilde{P}_{Z_{h-1}}(d\psi^{(\leq h)})$ is defined as $P_{Z_h}(d\psi^{(\leq h)})$ eq(21) with Z_{h-1} and σ_{h-1} replacing Z_h, σ_h, with

$$Z_{h-1}(\vec{k}') = Z_h(1 + C_h^{-1}(\vec{k}')z_h) \quad Z_{h-1}(\vec{k}')\sigma_{h-1}(\vec{k}') = Z_h(\sigma_h(\vec{k}') + C_h^{-1}(\vec{k}')s_h)$$

Moreover

$$\tilde{\mathcal{V}}^{(h)} = \mathcal{L}\tilde{\mathcal{V}}^{(h)} + (1 - \mathcal{L})\mathcal{V}^{(h)}$$

and

$$\mathcal{L}\tilde{\mathcal{V}}^{(h)} = \gamma^h n_h F_\nu^{(\leq h)} + (a_h - z_h)F_\alpha^{(\leq h)} + u_h F_U^{(\leq h)} \qquad (28)$$

The r.h.s of (27) can be written as

$$\int P_{Z_{h-1}}(d\psi^{(\leq h-1)}) \int \tilde{P}_{Z_{h-1}}(d\psi^{(h)}) e^{-\tilde{\mathcal{V}}^{(h)}(\sqrt{Z_h}\psi^{(\leq h)})} \qquad (29)$$

where $P_{Z_{h-1}}(d\psi^{(\leq h-1)})$ and $\tilde{P}_{Z_{h-1}}(d\psi^{(h)})$ are given by eq(21) with $Z_{h-1}(\vec{k}')$ replaced by $Z_{h-1}(0) \equiv Z_{h-1}$ and $C_h(\vec{k}')$ replaced with $C_{h-1}(\vec{k}')$ and $\tilde{f}_h^{-1}(\vec{k}')$ respectively, if

$$\tilde{f}_h(\vec{k}') = Z_{h-1}\left[\frac{C_h^{-1}(\vec{k}')}{Z_{h-1}(\vec{k}')} - \frac{C_{h-1}^{-1}(\vec{k}')}{Z_{h-1}}\right]$$

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and $\psi^{(\leq h)}$ replaced with $\psi^{(\leq h-1)}$ and $\psi^{(h)}$ respectively. Note that $\tilde{f}_h(\vec{k}')$ is a compact support function, with support of width $O(\gamma^h)$ and far $O(\gamma^h)$ from the "singularity" *i.e.* ωp_F. The Grassmanian integration $\tilde{P}_{Z_{h-1}}(d\psi^{(h)})$ has propagator

$$g^h_{\omega,\omega'}(\vec{x} - \vec{y}) = \int \tilde{P}_{Z_{h-1}}(d\psi^{(h)})\psi^-_{\omega,\vec{x}}\psi^+_{\omega',\vec{y}}$$

given by

$$\frac{1}{Z_{h-1}}\int d\vec{k}' e^{-i\vec{k}'(\vec{x}-\vec{y})}\frac{1}{A_{h-1}(k')}\tilde{f}_h(\vec{k}') \cdot$$
$$\begin{pmatrix} -ik_0 - \alpha(k') - v_0\sin k' & \sigma_{h-1}(k') \\ \sigma_{h-1}(k') & -ik_0 - \alpha(k') + v_0\sin k' \end{pmatrix} \tag{30}$$

where

$$A_h(\vec{k}') = [-ik_0 - \alpha(k')]^2 - (v_0\sin k')^2 - [\sigma_{h-1}(\vec{k}')]^2$$

It is convenient to write decompose the propagator as

$$g^{(h)}_{\omega,\omega}(\vec{x} - \vec{y}) = g^{(h)}_{L,\omega}(\vec{x} - \vec{y}) + C^{(h)}_2(\vec{x} - \vec{y}) \tag{31}$$

where

$$g^{(h)}_{L,\omega}(\vec{x} - \vec{y}) = \frac{1}{L\beta}\sum_{\vec{k}'}\frac{e^{-i\vec{k}'(\vec{x}-\vec{y})}}{-ik_0 + \omega v_0\sin k' + \alpha(k')}\tilde{f}_h(\vec{k}') \tag{32}$$

and, for any integer $N > 1$

$$|g^{(h)}_{L,\varepsilon\omega}(\vec{x} - \vec{y})| \leq \frac{\gamma^h C_N}{1 + (\gamma^h|\vec{x} - \vec{y}|)^N} \tag{33}$$

$$|C^{(h)}_2(\vec{x} - \vec{y})| \leq |\frac{\sigma_h}{\gamma^h}|^2\frac{\gamma^h C_N}{1 + (\gamma^h|\vec{x} - \vec{y}|)^N}$$

Moreover

$$|g^{(h)}_{\omega,-\omega}(\vec{x} - \vec{y})| \leq |\frac{\sigma_h}{\gamma^h}|\frac{\gamma^h C_N}{1 + (\gamma^h|\vec{x} - \vec{y}|)^N} \tag{34}$$

Finally we *rescale* the fields so that

$$\int P_{Z_{h-1}}(d\psi^{(\leq h-1)})\int \tilde{P}_{Z_{h-1}}(d\psi^{(h)})\, e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)})} \tag{35}$$

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so that
$$\mathcal{L}\hat{\mathcal{V}}^{(h)}(\psi) = \gamma^h \nu_h F_\nu^{(\leq h)} + \delta_h F_\alpha^{(\leq h)} + U_h F_U^{(\leq h)} \qquad (36)$$

where by definition
$$\nu_h = \frac{Z_h}{Z_{h-1}} n_h; \quad \delta_h = \frac{Z_h}{Z_{h-1}}(a_h - z_h); \quad U_h = (\frac{Z_h}{Z_{h-1}})^2 u_h \qquad (37)$$

We perform the integration
$$\int \tilde{P}_{Z_{h-1}}(d\psi^{(h)}) \, e^{-\hat{\mathcal{V}}^{(h)}(\sqrt{Z_{h-1}}\psi^{(\leq h)})} = e^{-\mathcal{V}^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\leq h-1)})} \qquad (38)$$

where $\mathcal{V}^{(h-1)}$ has the same form as $\mathcal{V}^{(h)}$ and the procedure can be iterated, as inserting (38) in (29) we have an expression like (20) with $h-1$ replacing h. The above procedure is iterated untill a scale h^* defined as the minimum h such that $\gamma^h > |\sigma_h|$. Then we will integrate directly the field $\psi^{(<h^*)} = \sum_{k=-\infty}^{h^*} \psi^{(k)}$ without splitting the corresponding integration in scales (as was done for $h > h^*$). This can be done as $g^{\leq h^*}(\vec{x} - \vec{y})$ verifies the bound eq(33) with h^* replacing h $i.e.$it verifies the bound valid for a single scale; the reason is that for momenta larger than $O(\gamma^{h^*})$ the theory is essentially a massless theory and for momenta smaller is a massive theory with mass $O(\gamma^{h^*})$. We will call $running$ $coupling$ $constants$ $\vec{v}_h = (U_h, \delta_h, \nu_h)$ and $renormalization$ $constants$ Z_h, σ_h; their behaviour as a function of h can be found by an iterative equation called $beta$ $function$ (see below).

2.2 Bounds for the effective potential

As explained, for example, in [BM], one can write the effective potential on scale j, if $h \leq j < 0$, as a sum of terms, which is indeed a finite sum for finite values of N, L, β. Each term of this expansion is associated with a $tree$ in the following way.

1) Let us consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $b \geq 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. n will be called the *order* of the unlabeled tree and the branching points will be called the *non trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order.

Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. It is then easy to see that the number of unlabeled trees with b end-points is bounded by 4^b.

We shall consider also the *labeled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

2) We associate a label $j \leq 0$ with the root and we denote $\mathcal{T}_{j,b}$ the corresponding set of labeled trees with n endpoints. Moreover, we introduce a family of vertical lines, labeled by an an integer taking values in $[j, 2]$, and we represent any tree $\tau \in \mathcal{T}_{j,n}$ so that, if v is an endpoint or a non trivial vertex, it is contained in a vertical line with index $h_v > h$, to be called the *scale* of v, while the root is on the line with index j. There is the constraint that, if v is an endpoint, $h_v > j + 1$.

The tree will intersect in general the vertical lines in set of points different from the root, the endpoints and the non trivial vertices; these points will be called *trivial vertices*. The set of the *vertices* of τ will be the union of the endpoints, the trivial vertices and the non trivial vertices. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

Moreover, there is only one vertex immediately following the root, which will be denoted v_0 and can not be an endpoint; its scale is $j + 1$.

3) With each endpoint v of scale $h_v \neq 2$ we associate one of the two local terms contributing to $\mathcal{L}\hat{\mathcal{V}}^{(h_v - 1)}$ in the r.h.s. of (36) and one space-time point $\vec{x}_v$. If $h_v = 2$ we associate one of the addend of $\mathcal{V}$ (9). We denote by V_a the set of end-points v with $h_v = 2$ to which is associated $e^{2in_v p_F x} \hat{\varphi}_{n_v}$ *i.e.*one of the addenda of P in (9); moreover $q = |V_a|$.

Moreover, we impose the constraint that, if v is an endpoint, $h_v = h_{v'} + 1$, if v' is the non trivial vertex immediately preceding v and $h_v \neq 1$. Given a vertex v, $\vec{x}_v$ denotes the family of all space-time points associated with one end-points following v.

4) If v is not an endpoint, the *cluster* L_v with frequency h_v is the set of endpoints following the vertex v; if v is an endpoint, it is itself a (*trivial*) cluster. The tree provides an organization of endpoints into a hierarchy of clusters.

5) We introduce a *field label f* to distinguish the field variables appearing in the terms associated with the endpoints as in item 3); the set of field labels associated with the endpoint v will be called I_v. Analogously, if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v; $\vec{x}(f)$, $\sigma(f)$ and $\omega(f)$ will denote the space-time point, the σ index and the ω index, respectively, of the field variable with label f.

6) We associate with any vertex v a set P_v with the labels associated with the *external fields* of v. These sets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \dots, v_{s_v}$ are the s_v vertices immediately following it, then $P_v \subset \cup_i P_{v_i}$. We shall denote Q_{v_i} the intersection of P_v and P_{v_i}; this definition implies that $P_v = \cup_i Q_{v_i}$. The subsets P_{v_i}/Q_{v_i}, whose union will be made, by definition, of the *internal fields* of v, have to be non empty, if $s_v > 1$, that is if v is a non trivial vertex. To each v is associated an integer N_v such that

$$\sum_{f \in P_v} \varepsilon(f) \vec{k}(f) = 2 N_v p_F$$

Given $\tau \in \mathcal{T}_{j,n}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with the previous constraints; let us call $\mathbf{P}$ one of this choices. Given $\mathbf{P}$, we consider the family $\mathcal{G}_{\mathbf{P}}$ of all connected Feynman graphs, such that, for any $v \in \tau$, the internal fields of v are paired by propagators of scale h_v, so that the following condition is satisfied: for any $v \in \tau$, the subgraph built by the propagators associated with all vertices $v' \geq v$ is connected. The sets P_v have, in this picture, the role of the external legs of the subgraph associated with v. The graphs belonging to $\mathcal{G}_{\mathbf{P}}$ will be called *compatible with $\mathbf{P}$* and we shall denote $\mathcal{P}_\tau$ the family of all choices of $\mathbf{P}$ such that $\mathcal{G}_{\mathbf{P}}$ is not empty.

As explained in detail in [BM], we can write, if $h \leq j \leq -1$,

$$\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) + L\beta \tilde{E}_{j+1} =$$
$$= \sum_{b=1}^{\infty} \sum_{\tau \in \mathcal{T}_{j,b}} \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sqrt{Z_j}^{|P_{v_0}|} \int d\vec{x}_{v_0} \tilde{\psi}^{[h,j]}(P_{v_0}) K_{\tau,\mathbf{P}}^{(j+1)}(\vec{x}_{v_0}) \qquad (39)$$

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where

$$\tilde{\psi}^{[h,j]}(P_v) = \prod_{f \in P_v} \psi^{[h,j]\sigma(f)}_{\vec{x}(f),\omega(f)}$$

and $K^{(j+1)}_{\tau,\mathbf{P}}(\vec{x}_{v_0})$ is a suitable function, which is obtained by summing the values of all the Feynman graphs compatible with $\mathbf{P}$, see item 6) above, and applying iteratively in the vertices of the tree, different from the endpoints and v_0, the $\mathcal{R}$-operation, starting from the vertices with higher scale; see again [BM] for a more precise definition.

In order to control, uniformly in L and β, the various sums in (39), one has to exploit in a careful way the $\mathcal{R}$ operation acting on the vertices of the tree, as explained in full detail in [BM]. The result of this analysis, which can be applied without any problem to the model studied in this paper, is a general bound which has a simple dimensional interpretation and can be easily generalized to bound the density Fourier transforms, as we shall prove in the following sections.

Before giving a brief description of this representation, let us see what happens if we erase the $\mathcal{R}$ operation in all the vertices of the tree. In this case one gets the bound

$$\int d\vec{x}_{v_0} |K^{(j+1)}_{\tau,\mathbf{P}}(\vec{x}_{v_0})| \leq \tag{40}$$

$$L\beta \, (C\bar{\varepsilon})^{b-q} |\sigma|^q \gamma^{-j(-2+|P_{v_0}|/2)} \prod_{\text{v not e.p}} \left(\frac{Z_{h_v}}{Z_{h_v-1}}\right)^{\frac{|P_v|}{2}} \gamma^{-(-2+\frac{|P_v|}{2}+\frac{\tilde{z}(P_v)}{2})} \prod_{v \in V_a} \frac{|\hat{\varphi}_{n_v}|}{|\sigma|}$$

where C is a suitable constant and $\bar{\varepsilon} = \max_{j+1 \leq j' \leq 0} |\vec{v}_{j'}|$; moreover $\tilde{z}(P_v) = 1$ if $|P_v| = 2$, $\omega(f_1) = -\omega(f_2)$ and $\sum_{f \in P_v} \varepsilon(f)\omega(f) + 2N_v p_F = 0$. The factor $\gamma^{\tilde{z}(P_v)}$ derives from the following inequality (see 3.106 of [BM]) if $k > h$

$$\frac{|\sigma_k|}{\gamma^k} \leq \frac{|\sigma_h|}{\gamma^h} \gamma^{\frac{1}{2}(h-k)} \leq \gamma^{\frac{1}{2}(h-k)}. \tag{41}$$

The good dependence on the number of end-points b derives from the use of the *Gram-Hadamard* inequality for the determinants appearing in $K^{(j+1)}_{\tau,\mathbf{P}}(\vec{x}_{v_0})$, see [BM]. The bound (41) allows to associate a factor $\gamma^{2-|P_v|/2+\frac{\tilde{z}(P_v)}{2}}$ with any trivial or non trivial vertex of the tree. This would allow to control the sums over the scale labels and $\mathcal{P}_\tau$, if $|P_v|$ were larger than 4 in all vertices, which is not true. The effect of the $\mathcal{R}$ operation is to improve the bound, so that there is a factor less than 1 associate even with the vertices where

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$|P_v|$ is equal to 2 or 4 and eq(23) holds. In order to explain how this works, we need a more detailed discussion of the $\mathcal{R}$ operation. We can consider for istance the case of $|P_v| = 4$; then

$$\mathcal{L} \int d\underline{\vec{x}} W(\underline{\vec{x}}) \prod_{i=1}^{4} \psi_{\vec{x}_i,\omega_i}^{[h,j]\sigma_i} = \int d\underline{\vec{x}} W(\underline{\vec{x}}) \prod_{i=1}^{4} \psi_{\vec{x}_4,\omega_i}^{[h,j]\sigma_i}] , \qquad (42)$$

where $\underline{\vec{x}} = (\vec{x}_1, \ldots, \vec{x}_4)$ and $W(\underline{\vec{x}})$ is the Fourier transform of $\hat{W}_{4,\vec{\omega}}^{(j)}(\vec{k}_1, \vec{k}_2, \vec{k}_3)$. Note that $W(\underline{\vec{x}})$ is translation invariant (despite the theory is not translationally invariant); hence $\psi_{\vec{x}_4,\omega_i}^{[h,j]\sigma_i}$ in the r.h.s. of (42) can be substituted with $\psi_{\vec{x}_k,\omega_i}^{[h,j]\sigma_i}$, $k = 1, 2, 3$ and we have four equivalent representations of the localization operation, which differ by the choice of the *localization point.*

We have

$$\mathcal{R} \int d\underline{\vec{x}} W(\underline{\vec{x}}) \prod_{i=1}^{4} \psi_{\vec{x}_i,\omega_i}^{[h,j]\sigma_i} = \int d\underline{\vec{x}} W(\underline{\vec{x}}) \left[\prod_{i=1}^{4} \psi_{\vec{x}_i,\omega_i}^{[h,j]\sigma_i} - \prod_{i=1}^{4} \psi_{\vec{x}_4,\omega_i}^{[h,j]\sigma_i} \right] =$$

$$= \int d\underline{\vec{x}} W(\underline{\vec{x}}) \Big[\psi_{\vec{x}_1,\omega_1}^{[h,j]\sigma_1} \psi_{\vec{x}_2,\omega_2}^{[h,j]\sigma_2} D_{\vec{x}_3,\vec{x}_4,\omega_3}^{1[h,j]\sigma_3} \psi_{\vec{x}_4,\omega_4}^{[h,j]\sigma_4} + \psi_{\vec{x}_1,\omega_1}^{[h,j]\sigma_1} D_{\vec{x}_2,\vec{x}_4,\omega_2}^{1[h,j]\sigma_2} \psi_{\vec{x}_4,\omega_3}^{[h,j]\sigma_3} \psi_{\vec{x}_4,\omega_4}^{[h,j]\sigma_4} +$$

$$+ D_{\vec{x}_1,\vec{x}_4,\omega_1}^{1[h,j]\sigma_1} \psi_{\vec{x}_4,\omega_2}^{[h,j]\sigma_2} \psi_{\vec{x}_4,\omega_3}^{[h,j]\sigma_3} \psi_{\vec{x}_4,\omega_4}^{[h,j]\sigma_4} \Big] \qquad (43)$$

where (again if $L = \beta = \infty$)

$$D_{\vec{y},\vec{x},\omega}^{1[h,j]\sigma} = \psi_{\vec{y},\omega}^{[h,j]\sigma} - \psi_{\vec{x},\omega}^{[h,j]\sigma}$$

The field $D_{\vec{y},\vec{x},\omega}^{1[h,j]\sigma}$ is dimensionally equivalent to the product of $|\vec{y} - \vec{x}|$ and the derivative of the field, so that the bound of its contraction with another field variable on a scale $j' < j$ will produce a "gain" $\gamma^{-(j-j')}$ with respect to the contraction of $\psi_{\vec{y},\omega}^{[h,j]\sigma}$. On the other hand, each term in the r.h.s. of (43) differs from the term which $\mathcal{R}$ acts on mainly because one $\psi^{[h,j]}$ field is substituted with a $D^{1[h,j]}$ field and some of the other $\psi^{[h,j]}$ fields are "translated" in the localization point. All three terms share the property that the field whose $\vec{x}$ coordinate is equal to the localization point is not affected by the action of $\mathcal{R}$. Analogous consideration can be done for the case $2n = 2$; note that one gets the factor $\gamma^{2(j-j')}$ if $\omega_1 = \omega_2$ and $\gamma^{(j-j')}$ if $\omega_1 = \omega_2$. One obtains a new representation of the effective potential, in place

of (39), such that

$$\mathcal{RV}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) = \sum_{b=1}^{\infty} \sum_{\tau \in \mathcal{T}_{j,b}} \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sum_{\alpha \in A_{\tau,\mathbf{P}}} \sqrt{Z_j}^{|P_{v_0}|} \int d\vec{x}_{v_0} D_\alpha \tilde{\psi}^{[h,j]}(P_{v_0}) K_{\tau,\mathbf{P},\alpha}^{(j+1)}(\vec{x}_{v_0})$$

(44)

where $A_{\tau,\mathbf{P}}$ labels a finite set of different terms, of counting power C^n, and, for any $\alpha \in A_{\tau,\mathbf{P}}$, D_α denotes an operator dimensionally equivalent to a derivative of order m_α. The important property of (44) is that

$$\int d\vec{x}_{v_0} |K_{\tau,\mathbf{P},\alpha}^{(j+1)}(\vec{x}_{v_0})| \leq L\beta \, (C\bar{\varepsilon})^{b-q} |\sigma|^q$$

(45)

$$\gamma^{-j(-2+|P_{v_0}|/2+m_\alpha)} \prod_{v \text{ not e.p}} (\frac{Z_{h_v}}{Z_{h_v-1}})^{|P_v|/2} \gamma^{-[-2+|P_v|/2+z(P_v)+\frac{\tilde{z}(P_v)}{2}]} \prod_{v \in V_a} \frac{|\hat{\varphi}_{n_v}|}{|\sigma|}$$

where $m_\alpha \geq z(P_{v_0})$ and $z(P_v)$ is vanishing unless
   1)$z(P_v) = 1$ if $|P_v| = 4$ and (23) is verified
   2)$z(P_v) = 2$ if $|P_v| = 2$ and $\omega_1 = \omega_2$ and $z(P_v) = 1$ if $|P_v| = 2$ and $\omega_1 = -\omega_2$ and (23) is verified.
   We will need in the following the following result (proved in [BGPS], [GS], [BoM]).
   Theorem. *There is a constant ε_0, such that, if $|\lambda| \leq \varepsilon_0$, then, uniformly in the infrared cutoff,*

$$\lambda_j = \lambda + O(\lambda^2) \,, \qquad \delta_j = O(\lambda^2) \,, \qquad h \leq j \leq -1$$

(46)

by which we obtain that $\bar{\varepsilon} \leq C'|U|$. In order to sum over the trees we need that $-2+|P_v|/2+z(P_v)+\frac{\tilde{z}(P_v)}{2} > 0$, which is however *not* true. We define V_b the set of v not end-points such that $|P_v| = 2,4$ and $-2+|P_v|/2+z(P_v)+\frac{\tilde{z}(P_v)}{2} \leq 0$. For any $v \in V_b$ it holds that, by definition

$$|\sum_{f \in P_v} \varepsilon(f)\omega(f)p_F + 2N_v p_F||_T \geq \frac{2\pi}{Q}$$

(47)

so that for any $v \in V_b$

$$\gamma^{h_{v'}} \geq \frac{\pi}{2Q}$$

(48)

It holds that for any $v \in V_b$ there is at least an end-point $\bar{v} \in V_a$ in the cluster L_v. This is proved by contradiction: assume that in $v \in V_b$ there

are no end-points $\bar{v} \in V_a$; as the end-points v not belonging to V_a are such that $\sum_{f \in P_v} \varepsilon(f)\vec{k}'_f = 0$, where $\vec{k}'$ is the momentum measured from the Fermi surface, then a vertex v not containing end-points $\in V_b$ is such that $\sum_{f \in P_v} \varepsilon(f)\vec{k}'_f = 0$, which means $v \notin V_a$ as the l.h.s. of (47) is vanishing. We consider then a maximal $v \in V_b$, $i.e.v$ such that there is no $\bar{v} \in V_a$ in L_v; there is a path $\mathcal{C} \in \tau$ (called maximal path) connecting v with an end point $\bar{v} \in V_a$; then by (48)

$$1 = Q^2 Q^{-2} \leq Q^2 \prod_{v \in \mathcal{C}} \gamma^{-\hat{z}(P_v)} \tag{49}$$

where $\hat{z}(P_v) = 2$ if $v \in \mathcal{C}$ and zero otherwise. We consider now the next-to-maximal $v \in V_b$ $i.e.$not belonging to maximal paths and such that there is only one $\bar{v} \in V_b$ such that $\bar{v} < v$; again there is a next to maximal path $\mathcal{C} \in \tau$ connecting v with an end point $\bar{v} \in V_a$ (not belonging to the maximal paths) and we obtain again a bound like (49). We proceed in this way untill all the $v \in V_b$ are on paths and at the end we get

$$1 \leq Q^{2q_\tau} \prod_{v \in V_b} \gamma^{-2}$$

so that

$$\int d\vec{x}_{v_0} |K^{(j+1)}_{\tau,\mathbf{P},\alpha}(\vec{x}_{v_0})| \leq L\beta \, (C\bar{\varepsilon})^{b-q}(Q^2|\sigma|)^q \tag{50}$$

$$\gamma^{-j(-2+|P_{v_0}|/2+m_\alpha)} \prod_{\text{v not e.p}} (\frac{Z_{h_v}}{Z_{h_v-1}})^{|P_v|/2} \gamma^{-[-2+|P_v|/2+z(P_v)+\frac{\bar{z}(P_v)}{2}+\hat{z}(P_v)]} \prod_{v \in V_a} \frac{|\hat{\varphi}_{n_v}|}{|\sigma|}$$

where $\hat{z}(P_v) = 2$ if $v \in V_b$. Proceding then as in [BM] the sum over τ is convergent provided that

$$|\sqrt{\sigma}Q^2| \leq \varepsilon \tag{51}$$

Remark: The above argument does not interfere in the determinant estimates, which has to be performed in the coordinate space. In fact as all the terms in the truncated expectations not verifying (48) are exactly vanishing we can "freely" insert in the sums (before performing the bounds) a χ-function ensuring (48) (see [BM]).

2.3 Bounds for the density

We start from the generating function and we perform iteratively the integration of the ψ variables; after the fields $\psi^{(0)}, ...\psi^{(j+1)}$ have been integrated, we can write

$$e^{\mathcal{W}(\phi,J)} = e^{-L\beta E_j} \int P_{\tilde{Z}_j, C^{\varepsilon}_{h,j}}(d\psi^{[h,j]}) e^{-\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}) + \mathcal{B}^{(j)}(\sqrt{Z_j}\psi^{[h,j]}, \phi, J)}$$

where $\mathcal{B}^{(j)}(\sqrt{Z_j}\psi, \phi, J)$ denotes the sum over the terms containing at least one ϕ or J field; we shall write it in the form

$$\mathcal{B}^{(j)}(\sqrt{Z_j}\psi, \phi) = \mathcal{B}^{(j)}_{\phi}(\sqrt{Z_j}\psi) + W^{(j)}_R(\sqrt{Z_j}\psi, \phi, J)$$

where $\mathcal{B}^{(j)}_{\phi}(\psi)$ denote the sums over the terms containing only one ϕ.

In order to control the Schwinger functions expansion, we have to suitably regularize $\mathcal{B}^{(j)}_{\phi}(\sqrt{Z_j}\psi)$. We want to show that, if $j \leq -1$, it can be written in the form

$$\mathcal{B}^{(j)}_{\phi}(\sqrt{Z_j}\psi) = \sum_{\omega} \sum_{i=j+1}^{0} \int d\vec{x} d\vec{y} \;\cdot$$

$$\left[\phi^{+}_{\vec{x},\omega} g^{Q,(i)}_{\omega}(\vec{x} - \vec{y}) \frac{\partial}{\partial \psi^{+}_{\vec{y}\omega}} \mathcal{V}^{(j)}(\sqrt{Z_j}\psi) + \frac{\partial}{\partial \psi^{-}_{\vec{y},\omega}} \mathcal{V}^{(j)}(\sqrt{Z_j}\psi) g^{Q,(i)}_{\omega}(\vec{y} - \vec{x}) \phi^{-}_{\vec{x},\omega}\right] +$$

$$+ \sum_{\omega} \int \frac{d\vec{k}}{(2\pi)^2} \left[\hat{\psi}^{[h,j]+}_{\vec{k},\omega} \hat{Q}^{(j+1)}_{\omega}(\vec{k}) \hat{\phi}^{-}_{\vec{k},\omega} + \hat{\phi}^{+}_{\vec{k},\omega} \hat{Q}^{(j+1)}_{\omega}(\vec{k}) \hat{\psi}^{[h,j]-}_{\vec{k},\omega}\right] \tag{52}$$

where

$$\hat{g}^{Q,(i)}_{\omega}(\vec{k}) = \hat{g}^{(i)}_{\omega}(\vec{k}) \hat{Q}^{(i)}_{\omega}(\vec{k})$$

and $Q^{(j)}_{\omega}(\vec{k})$ is defined inductively by the relations

$$\hat{Q}^{(j)}_{\omega}(\vec{k}) = \hat{Q}^{(j+1)}_{\omega}(\vec{k}) - z_j Z_j D_{\omega}(\vec{k}) \sum_{i=j+1}^{0} \hat{g}^{Q,(i)}_{\omega}(\vec{k}) , \quad \hat{Q}^{(0)}_{\omega}(\vec{k}) = 1$$

The second line in (52) has a simple interpretation in terms of Feynman graphs; it is obtained by taking all the graphs contributing to $\mathcal{V}^{(j)}(\sqrt{Z_h}\psi)$ and, given a single graph, by adding a new space-time-point $\vec{x}$ associated with a term $\phi_{\vec{x}}\psi_{\vec{x}}$ and contracting the correspondent ψ field with one of the

external fields of the graph through a propagator $\sum_{i=j+1}^{0} g_{\omega}^{Q,(i)}(\vec{x}-\vec{y})$. Hence, it is very easy to see that (52) is satisfied for $j = -1$. The fact that it is valid for any j follows from our choice to regularize $\mathcal{B}_{\phi}^{(j)}(\sqrt{Z_j}\psi)$ by regularizing the effective potential in the r.h.s. of (52), that is by decomposing $\mathcal{V}^{(j)}$ as $\mathcal{L}\mathcal{V}^{(j)} + \mathcal{R}\mathcal{V}^{(j)}$. This implies, in particular, that we have to extract from the effective potential the term proportional to z_j; the corresponding contribution to $\mathcal{B}_{\phi}^{(j)}(\sqrt{Z_j}\psi)$ is then absorbed in the term in the third line of (52), before rescaling the field ψ and performing the integration of the scale j field. It is then easy to check that (52) is satisfied for $j = \bar{j} + 1$, if it is satisfied for $j = \bar{j}$ (for more details, see [BM]).

Note that, if $\hat{g}_{\omega}^{(j)}(\vec{k}) \neq 0$

$$\hat{Q}_{\omega}^{(j)}(\vec{k}) = 1 - z_j f_{j+1}^{\varepsilon}(\vec{k})\frac{Z_j}{\tilde{Z}_j(\vec{k})}$$

Hence, the propagator $\hat{g}_{\omega}^{Q,(i)}(\vec{k})$ is equivalent to $\hat{g}_{\omega}^{(i)}(\vec{k})$, as concerns the dimensional bounds.

We can expand the functional in terms of trees, as we did for the effective potential, by suitably modifying the definitions adding a new type of end-points, to be called of type ϕ and to which associated the terms in the last line of (52). Moreover we change the definition of the sets P_v so that the set P_v includes both the field variables of type ψ which are not yet contracted in the vertex v, to be called *normal external fields*, and those which belong to a normal endpoint and are contracted with a field variable belonging to an endpoint of type ϕ through a propagator $g^{Q,(h_v)}$, to be called *special external fields* of v.

From the expansion of the two point Schwinger function we obtain an expansion for the density. We call h_1 and h_2, with $h_1 \leq h_2$ (say) the scales of the external propagators $g^{Q,(h_v)}$; we call moreover $\bar{v}$ the first vertex such that both the two end-points of type ϕ are connected; of course $\gamma^{h_{\bar{v}}} \leq \gamma^{h_1}$. With respect to the expansion for the effective potential, $\mathcal{R} = 1$ for $v \leq \bar{v}$ with $|P_v| = 4$. We write

$$\rho_n = \int d\vec{k}\rho_n(\vec{k}) = \sum_{h_1,h_2} \int d\vec{k} f_{h_1}(\vec{k}) f_{h_2}(\vec{k}+2np_F)\rho_n(\vec{k}) \tag{53}$$

ρ_n is given by a sum over trees; in order to sum over the scales of the vertices of the tree we sum over the difference $h_v - h_{v'}$ between consecutive vertices

and over the scale of a vertex; instead of choosing v_0 like in the preceding section we fix $\bar{v}$. Dimensionally the bound for (53) is very similar to the bound for the effective potential with two external line; the ϕ end-points in (53) are like two ν-vertices, and the integration over $\vec{k}$ allows to associate to the ϕ end points the dimensional factor $\gamma^{h_1}\gamma^{h_2}$. By can bound then (53) by an expression similar to (46), with a total dimensional factor γ^j (without $L\beta$ factor). However $\mathcal{R} = 1$ between $\bar{v}$ and v_0 so that we cannot sum over the scale assignments. But we can write $\gamma^j = \gamma^j\gamma^{j-h_{\bar{v}}}\gamma^{-(j-h_{\bar{v}})}$ and at the end

$$|\rho_n| \leq \sum_{b=1}^{\infty}\sum_{q\leq b}\sum_{h_{\bar{v}}}[\gamma^{h_{\bar{v}}} \tag{54}$$

$$\sum_{\tau\in\mathcal{T}_{h_{\bar{v}},b,q}} (C\bar{\varepsilon})^{b-q}(Q^2|\sigma|)^q \prod_{\text{v not e.p}} (\frac{Z_{h_v}}{Z_{h_v-1}})^{|P_v|/2}\gamma^{-D_v} \sum_{\{n_v\}}\prod_{v\in V_a} \frac{|\hat{\varphi}_{n_v}|}{|\sigma|}$$

where

$$D_v = -[-2 + |P_v|/2 + z(P_v) + \frac{\tilde{z}(P_v)}{2} + \hat{z}(P_v) + z'(P_v)]$$

and $z'(P_v) = \frac{1}{3}$ for $\bar{v} \leq v \leq v_0$ and zero otherwise. As $D_v > 0$ the second line of (54) is surely bounded by a constant. There are at most $4b$ non diagonal propagators so that

$$|n| \leq 4b + 1 + \sum_{v\in V_a} |n_v|$$

so that there exists a $v^* \in V_a$ such that $|n_{v^*}| \geq C\frac{|n|}{b}$ so that the sum over $\{n_v\}$ gives

$$|\rho_n| \leq \sum_{b=1}^{\infty}\sum_{q\leq b}\sum_{h_{\bar{v}}}\gamma^{h_{\bar{v}}} \tag{55}$$

$$(\frac{b}{|n|})^N \sum_{\tau\in\mathcal{T}_{h_{\bar{v}},b,q}} (C\bar{\varepsilon})^{b-q}(Q^2|\sigma|)^q \prod_{\text{v not e.p}} (\frac{Z_{h_v}}{Z_{h_v-1}})^{|P_v|/2}\gamma^{-D_v}$$

2.4 The case $|n| > 1$

2.4.1 The case $q \geq 2$.

By (55) we get

$$\sum_{b=2}^{\infty}\sum_{q=2}^{\infty}\varepsilon^{b-q}(Q^2|\sigma|^{\frac{1}{2}})^q|\sigma|^{\frac{q}{2}}(\frac{\kappa}{|n|})^N \leq |\sigma|\sum_{b=2}^{\infty}\varepsilon^b(\frac{b}{|n|})^N \leq \frac{C_N\varepsilon^2|\sigma|}{|n|^N} \tag{56}$$

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2.4.2 The case $q = 1$ with at least a non diagonal propagator

If $q = 1$ and there is at least a non diagonal propagator (such terms are contributing to $\tilde{\rho}_n$) we can improve the bound (55) with a factor $\frac{\sigma_h}{\gamma^h}$, by (31). Then we get for $b \geq 2$ (so that there is at least a vertex λ, δ, ν and using Theorem 1)

$$\sum_{h_{\bar{v}}} \gamma^{h_{\bar{v}}} \frac{\sigma_{h_{\bar{v}}}}{\gamma^{h_{\bar{v}}}} \sum_{b=1}^{\infty} |U|^{b-1} (Q^2 |\sigma|^{\frac{1}{2}})^1 |\sigma|^{\frac{1}{2}} (\frac{b}{|n|})^N$$

$$\leq |U| \sum_{h_{\bar{v}}} \sigma_{h_{\bar{v}}} \sqrt{|\sigma|} \sum_{b=0}^{\infty} \varepsilon^b (\frac{b}{|n|})^N \leq \frac{C_N |\sigma|}{|n|^N}$$

using that (which will be proved in §3 below)

$$\sum_{k=0}^{h^*} |\sigma_k| \leq |\frac{1}{\eta} |\sigma| (|\sigma|^{-\eta} - 1)|$$

We can write explicitly the contribution with $\kappa = q = 1$

$$\hat{\rho}_n^{1,1} = \sum_{h,h'=h^*}^{1} \frac{1}{Z_h Z_{h'}} \sum_{m \neq 0, n} \sum_{\substack{\omega_1, \omega_1' \\ \omega_2, \omega_2'}} \delta_{2m - \omega_1 + \omega_1' - \omega_2 + \omega_2', 2n} \tag{57}$$

$$\lambda \hat{\varphi}_m \int d\vec{k}' g_{\omega_1,\omega_1'}^{(h)}(\vec{k}') g_{\omega_2,\omega_2'}^{(h')}(\vec{k}' + (2m + \omega_1' - \omega_2) p_F)$$

where $g^{(1)}(\vec{k}) \equiv g_{\omega,\omega}^{(1)}(\vec{k}')$ if $\vec{k} = \vec{k}' + \omega p_F$. We find the following bound

$$\sum_{h,h'=h^*}^{1} \frac{1}{Z_h Z_{h'}} | \int d\vec{k}' g_{\omega_1,-\omega_1}^{(h)}(\vec{k}') g_{\omega_2,\omega_2'}^{(h')}(\vec{k}' + (2m + \omega_1' - \omega_2) p_F)| \leq$$

$$C \sum_{h=h^*}^{1} [|\sigma_h \gamma^{-h}|^2 + |\sigma_h \gamma^{-h}|] \leq C' \tag{58}$$

This completes the prof of the first of (16).

2.4.3 The case $q = 1$ with no non diagonal propagator

Such terms are contributing to c_n. We can derive such terms by the following functional integral

$$\frac{\partial}{\partial J} \int P(d\psi) e^{\mathcal{V} + (\psi,\phi) + \int J\lambda \hat{\varphi}_n e^{inx} \psi^+ \psi^-}|_{J=0} \tag{59}$$

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with $|n| \neq 1$. By performing the functional integration it appears that in the effective potential there are terms of the form

$$J \int d\vec{x} d\vec{x}_1 d\vec{x}_2 w(\vec{x} - \vec{x}_1, \vec{x} - \vec{x}_2) \lambda \hat{\varphi}_n e^{i2np_F x} \psi_{\vec{x}_1}^+ \psi_{\vec{x}_2}^+$$

which are dimensionally marginal (the J-vertex is dimensionally as a couple of external ψ-fields). In each tree there is a path $\mathcal{C}$ connecting the end-point associated to J with the root; each $v \in \mathcal{C}$ can have dimension greater or *equal* to zero and we call $h_{\bar{v}}$ is the maximal cluster with two external lines, whose external momenta verify

$$|\sum_{i=1}^{2} \varepsilon_i \vec{k}_i'| = |\sum_i \varepsilon_i p_F + 2np_F| \tag{60}$$

The r.h.s. of the above equation cannot be vanishing and using that $\gamma^{-h_{\bar{v}}} \leq Q$ then

$$\sum_{h_1 \geq h_2 \geq ... h^*} \leq \frac{(\log Q)^q}{q!}$$

and

$$\sum_{q=1}^{\infty} U^{q-1} \frac{(\log Q)^q}{q!} \leq \log Q Q^{CU}$$

The first order contribution is given by

$$c_n^{(1)}(\sigma) = \int d\vec{k} \Big\{ \tilde{g}_{1,1}^{(1)}(\vec{k}') \, \tilde{g}_{1,1}^{(1)}(\vec{k}' + 2n\vec{p}_F) \tag{61}$$

$$+ \sum_{\omega=\pm 1} \Big[\tilde{g}_{1,1}^{(1)}(\vec{k}') \sum_{h=h^*}^{0} \frac{1}{Z_h} \tilde{g}_{\omega,\omega}^{(h)}(\vec{k}' + 2n\vec{p}_F + (1-\omega)\vec{p}_F)$$

$$+ \sum_{h=h^*}^{0} \frac{1}{Z_h} \tilde{g}_{\omega,\omega}^{(h)}(\vec{k}') \, \tilde{g}_{1,1}^{(1)}(\vec{k}' + 2n\vec{p}_F - (1-\omega)\vec{p}_F)$$

$$+ \sum_{h=h^*}^{0} \frac{1}{Z_h} \tilde{g}_{\omega,-\omega}^{(h)}(\vec{k}') \sum_{h'=h^*}^{0} \frac{1}{Z_{h'}} \tilde{g}_{-\omega,\omega}^{(h')}(\vec{k}' + 2n\vec{p}_F)$$

$$+ \sum_{h=h^*}^{0} \frac{1}{Z_h} \tilde{g}_{\omega,\omega}^{(h')}(\vec{k}') \sum_{h'=h^*}^{0} \frac{1}{Z_{h'}} \tilde{g}_{\omega,\omega}^{(h')}(\vec{k}' + 2n\vec{p}_F) +$$

$$\sum_{h=h^*}^{0} \frac{1}{Z_h} \tilde{g}_{\omega,\omega}^{(h)}(\vec{k}') \sum_{h'=h^*}^{0} \frac{1}{Z_{h'}} \tilde{g}_{-\omega,-\omega}^{(h')}(\vec{k}' + (2n+2\omega)\vec{p}_F) \Big] \Big\}$$

It is easy to see that for the integrals in the first four lines of (61) are bounded by a constants, as there is at least a non diagonal propagator or an ultraviolet one; for the fifth integral the bound is

$$\sum_{h=h^*}^{0} | \int d\vec{k}' \frac{1}{Z_h} \tilde{g}_{\omega,\omega}^{(h')}(\vec{k}') \sum_{h'=h^*}^{0} \frac{1}{Z_{h'}} \tilde{g}_{\omega,\omega}^{(h')}(\vec{k}' + 2n\vec{p}_F)| \le$$

$$\sum_{h=h^*}^{0} | \int d\vec{k}' \tilde{g}_{\omega,\omega}^{(h')}(\vec{k}') \sum_{h'=h^*}^{0} \tilde{g}_{\omega,\omega}^{(h')}(\vec{k}' + 2n\vec{p}_F)| \le logQ \qquad (62)$$

If fact if $h < h'$ and noting that $|h'| \le \log Q$ we find

$$\sum_{h<h'} \gamma^{2h} \gamma^{-h} \gamma^{-h'} = \sum_{|h'|\le \log Q} = \log Q$$

This proves the second of (16).

2.5   The case $|n| = 1$

2.5.1   The case $q = 0$

The main difference with respect to the case $|n| > 1$ is that there are terms with $q = 0$ and containing at least one non diagonal propagator (such kind of contribution is of course absent if $|n| > 1$ by momenta conservation). The case $q = \kappa = 0$ gives

$$c_1(\sigma) = \int d\vec{k} \Big\{ \sum_{h=h^*}^{0} \frac{\tilde{g}_{-1,1}^{(h)}(\vec{k}')}{\sigma} + \tilde{g}_{1,1}^{(1)}(\vec{k}') \, \tilde{g}_{1,1}^{(1)}(\vec{k}' + 2\vec{p}_F) \qquad (63)$$

$$+ \sum_{\omega=\pm 1} \Big[\tilde{g}_{1,1}^{(1)}(\vec{k}') \sum_{h=h^*}^{0} \frac{1}{Z_h} \tilde{g}_{\omega,\omega}^{(h)}(\vec{k}' + (3 - \omega)\vec{p}_F) +$$

$$\sum_{h=h^*}^{0} \frac{1}{Z_h} \tilde{g}_{\omega,\omega}^{(h)}(\vec{k}') \, \tilde{g}_{1,1}^{(1)}(\vec{k}' + (1 + \omega)\vec{p}_F) \Big] \Big\}$$

$$= -F(\sigma, L, U) + \tilde{c}_1(\sigma)$$

and more explicitly

$$F(\sigma, \lambda, U) \equiv \sum_{h=h^*}^{0} \frac{\tilde{g}_{-1,1}^{(h)}(\vec{k}')}{\sigma} = \qquad (64)$$

$$\sum_{h=h^*}^{0} \int d\vec{k} \frac{1}{Z_h} \frac{\sigma_h}{\sigma} \frac{f_h(\vec{k})}{k_0^2 + \sin^2 k' + (1 - \cos k')^2 + \sigma_h^2}$$

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where σ_h, Z_h verify the flow equation (77) below, and $|\tilde{c}_1| \leq C$ where C is a constant. In bounding $q = 0$, $\kappa \geq 1$ and at least a non-diagonal propagator we use (55) with the improving that there is at least a $\frac{\sigma_{h\bar{v}}}{\gamma^{h\bar{v}}}$ more for the presence of the non diagonal propagator; we get

$$|\rho_n| \leq \sum_{\kappa=1}^{\infty} \sum_{q \leq k} \sum_{h_{\bar{v}}} |\sigma_{h_{\bar{v}}}| \tag{65}$$

$$\gamma^{-h_{\bar{v}}} \sum_{\tau \in \mathcal{T}_{h_{\bar{v}}, \kappa, q}} (C\bar{\varepsilon})^{\kappa-q} (Q^2 |\sigma|)^q \prod_{\text{v not e.p}} \left(\frac{Z_{h_v}}{Z_{h_v-1}}\right)^{|P_v|/2} \gamma^{-D_v} \sum_{\{n_v\}} \prod_{v \in V_a} \frac{|\hat{\varphi}_{n_v}|}{|\sigma|}$$

$$\leq C_N |U| \frac{|\sigma|}{\eta} [|\sigma|^{-\eta} - 1]|$$

2.5.2 The case $q \geq 1$

If $q \geq 2$ we proceed as in section 2.4.1 and we get the bound $C_N |\sigma|$. Again, if $q = 1$ and there is at least a non diagonal propagator we get the bound $C_N |\sigma|$ proceding as in 2.4.2. Finally if $q = 1$ and there are no non diagonal propagators we proceed as in 2.4.3 but the clusters v containing $\lambda \hat{\varphi}_{\pm 1}$ with two or for external lines are such that $N_v = 2$ so that $\gamma^{-h_{v'}} \leq \frac{\pi}{4}$; we proceed then as in 2.4.3 but instead of Q in the bounds we have an harmless constant and again we get the bound $C_N |\sigma|$.

Then by (64) can be written as

$$\frac{\lambda^2}{\eta} [(\frac{|\sigma|}{A})^{-\eta} - 1][a^{-1} + F(\lambda, U, \sigma)] + \sigma f(\lambda, U, \sigma) = 1 \tag{66}$$

with $|F| \leq C|U|$, $|f| \leq C$ and $\eta_1 = \beta_1 U + \tilde{\eta}_1$, $\eta_2 = \beta_1 U + \tilde{\eta}_2$, $|\tilde{\eta}_1|, |\tilde{\eta}_2| \leq CU^2$, and C, a, β_1, A are positive constants.

The term F in the above equation comes just from from the terms with $|V| = 1$, $q \geq 2$ and no non diagonal propagators. In the case the only vertex belonging to $V(\tau)$ is $\lambda \hat{\varphi}_1$ and it is contracted at scale $k = 1$; by using

$$\frac{\sigma_1}{\gamma^1} \leq \gamma^{\frac{1}{2}(h-1)} \frac{\sigma_h}{\gamma^h}$$

we see that we can renormalize all the clusters between 1 and h and containing $\lambda \hat{\varphi}_1$, so that we have a bound of the form, for $q \geq 1$

$$|\rho_1^{q,1}(\tau)| \leq (C\bar{\varepsilon})^{q-1} \frac{\sigma_h}{\gamma^h} \gamma^h \prod_{\text{vnote.p.}} \gamma^{-(-2+|P_v|+z(P_v)+\bar{z}(P_v))} \tag{67}$$

which gives the bound for F.

3 The flow of the running coupling constants

3.1 The Beta function

The equations for the running coupling constants are, for $h \geq h^*$

$$\nu_{h-1} = \gamma \nu_h + G_\nu^h \quad U_{h-1} = U_h + G_U^h$$
$$\sigma_{h-1} = \sigma_h + G_\sigma^h \quad \delta_{h-1} = \delta_h + G_\delta^h$$
$$\frac{Z_{h-1}}{Z_h} = 1 + G_z^h \tag{68}$$

It is convenient to split $G_i^{(h)}$, with $i = \mu, \sigma, \nu$ as

$$G_i^h(\mu_h, \nu_h, \sigma_h; ...; \mu_0, \nu_0, \sigma_0) = \tag{69}$$
$$G_i^{1,h}(\mu_h, \nu_h; ...; \mu_0, \nu_0) + G_i^{2,h}(\mu_h, \nu_h, \sigma_h; ...; \mu_0, \nu_0, \sigma_0)$$

where we have splitted $g_{\omega,\omega}^{(h)}$ as in eq(31) and $G_i^{1,h}$ contains no non diagonal propagators and only the part $g_{L;\omega,\omega}^{(h)}$ of the diagonal propagators; moreover there are no vertices with $m \neq 0$; in $G_i^{2,h}$ there are all the remaining contributions. It is easy to check that for $i = \mu, \nu$, for $max_{k \geq h}|\vec{v}_k| \leq \varepsilon$

$$|G_i^{2,h}| \leq C[\frac{\sigma_h}{\gamma^h}]^2 \varepsilon^2 \tag{70}$$

This follows from the bound eq(33) for C_2^h and from the fact that ν, μ are momentum conserving terms. For $i = \sigma$ by symmetry reasons, $G_i^{1,h} \equiv 0$ and

$$|G_\sigma^{2,h}(\mu_h, \nu_h, \sigma_h; ...; \mu_0, \nu_0, \sigma_0)| \leq C|U_h \sigma_h| \tag{71}$$

We decompose, if $i = \mu, \nu$

$$G_i^{1,h}(\mu_h, \nu_h; ...; \mu_0, \nu_0) = $$
$$\bar{G}_i^{1,h}(\mu_h; ...; \mu_0) + \hat{G}_i^{1,h}(\mu_h, \nu_h; ...; \mu_0, \nu_0) \tag{72}$$

where the first term in the r.h.s. of eq(72) is obtained putting $\nu_k = 0$, $k \geq h$ in the l.h.s. It is easy to see, from the fact that $g_{L,\omega}^{(h)}(\vec{x}; \vec{y})$ can be divided in a even part plus a correction smaller than a factor $\gamma^{\frac{h}{4}}$, for $max_{k \geq h}|\vec{v}_k| \leq \varepsilon$

$$|\bar{G}_\nu^{1,h}(\mu_h; ...; \mu_0) \leq C\varepsilon \gamma^{\frac{h}{4}} \tag{73}$$

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On the other hand

$$|\bar{G}_\mu^{1,h}(\mu_h;...;\mu_h)| \le C\varepsilon^2\gamma^{\frac{h}{2}} \tag{74}$$

as one can prove using the exact solution of the Luttinger model[**?**, **?**, **?**]. Moreover we have that, for $i = \nu, \mu$

$$|\hat{G}_i^{1,h}(\mu_h, \nu_h;...;\mu_0, \nu_0)| \le C\nu_h|U_h|^2 \tag{75}$$

Finally by a second order computation one obtains

$$G_\sigma^{1,h} = \sigma_h U_h[\beta_1 + \bar{G}_\sigma^{1,h}]$$
$$G_z^{1,h} = U_h^2[\beta_2 + \bar{G}_z^{1,h}] \tag{76}$$

with β_1, β_2 non vanishing positive constants and $|\bar{G}_\sigma^{1,h}| \le C|U_h|$, and $|\bar{G}_z^{1,h}| \le C|U_h|$.

By using the above properties we can control the flow of the running coupling constants. In fact, if $|\nu_k| \le C\varepsilon[\gamma^{\frac{k}{4}} + \frac{|\sigma_k|}{\gamma^k}]$ for any $k \ge h^*$ (what will be proved in the following section) it follows that there exist positive constants c_1, c_2, c_3, c_4, C such that, if λ, u are small enough and $h \ge h^*$:

$$|U_{h-1} - U| < CU^{3/2}$$
$$e^{-U\beta_1 c_3 h} < \frac{|\sigma_{h-1}|}{|\sigma_0|} < e^{-U\beta_1 c_4 h} \tag{77}$$
$$e^{-\beta_2 c_1 U^2 h} < Z_{h-1} < e^{-\beta_2 c_2 U^2 h}$$

3.1.1 Determination of the counterterm ν

We show finally that it is possible to fix ν to a λ-independent value; more exactly we show that it is possible to choose ν as in the $\lambda = 0$ case so that ν_k is small for any $k \ge h^*$. In the $\lambda = 0$ case $\sigma_k = 0$ and there are no contribution to the effective potential with $m \ne 0$; calling $\tilde{\nu}_k, \tilde{\mu}_k$ the analogous of ν_k, μ_k we can write

$$\tilde{\nu}_h = \gamma^{-h+1}[\nu + \sum_{k=h+1}^{1} \gamma^{k-2}G_\nu^{1,k}(\tilde{\nu}, \tilde{\mu})] \tag{78}$$

where $G_\nu^{1,k}(\tilde{\nu}, \tilde{\mu}) = G_\nu^{1,k}(\tilde{\nu}_k, \tilde{\mu}_k; ....; \tilde{\nu}_0, \tilde{\mu}_0)$. We choose

$$\nu = -\sum_{k=-\infty}^{1} \gamma^{k-2}G_\nu^{1,k}(\tilde{\nu}, \tilde{\mu}) \tag{79}$$

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then

$$\tilde{\nu}_h = -\gamma^{-h} \sum_{k=-\infty}^{h} \gamma^{k-1} G_\nu^{1,k}(\tilde{\nu}, \tilde{\mu}) \qquad (80)$$

and $|\tilde{\nu}_h| \leq C\varepsilon\gamma^{\frac{h}{4}}$ as by (73)

$$\gamma^{-h} \sum_{k=-\infty}^{h} \gamma^{k-1} |G_\nu^{1,k}(\tilde{\nu}, \tilde{\mu})| \leq C'\varepsilon\gamma^{-h} \sum_{k=-\infty}^{h} \gamma^k \gamma^{\frac{1}{4}k} \leq C\varepsilon\gamma^{\frac{h}{4}} \qquad (81)$$

For the model with $\lambda \neq 0$, for $h \geq h^*$

$$\nu_h = \gamma^{-h+1}[\nu + \sum_{k=h+1}^{1} \gamma^{k-2}[G_\nu^{2,k}(\nu, \mu, \sigma) + G_\nu^{1,k}(\nu, \mu)]] \qquad (82)$$

and inserting ν given by eq(79)

$$\nu_h - \tilde{\nu}_h = \gamma^{-h+1}\{ \sum_{k=h+1}^{1} \gamma^{k-2} G_\nu^{2,k}(\nu, \mu, \sigma)$$

$$+ \sum_{k=h+1}^{1} \gamma^{k-2}[G_\nu^{1,k}(\nu, \mu) - G_\nu^{1,k}(\tilde{\nu}, \tilde{\mu})]\} \qquad (83)$$

We prove that, for $h \geq h^*$

$$|\nu_h - \tilde{\nu}_h| \leq \varepsilon\bar{C}(\frac{\sigma_h}{\gamma^h})^2 \qquad |\tilde{\mu}_h - \mu_h| \leq \varepsilon(\frac{\sigma_h}{\gamma^h})^2 \qquad (84)$$

The proof is done by induction assuming that it holds for scales $\geq h+1$ and proving by (83) that it holds for scale h. Looking at the first sum in (83) and using (70),(77)

$$\gamma^{-h} \sum_{k=h+1}^{1} \gamma^{k-2}|G_\nu^{2,k}(\nu, \mu)| \leq C_1\gamma^{-h}\varepsilon^2 \sum_{k=h+1}^{1} \gamma^{k-2}(\frac{\sigma_k}{\gamma^k})^2$$

$$\leq (\frac{\sigma_h}{\gamma^h})^2 C_2\varepsilon^2 \sum_{k=h+1}^{1} \gamma^{h-k}(\frac{\sigma_k}{\sigma_h})^2 \leq C_3\varepsilon^2(\frac{\sigma_h}{\gamma^h})^2 \qquad (85)$$

Finally we can write

$$G_\nu^{1,k}(\nu, \mu) - G_\nu^{1,k}(\tilde{\nu}, \tilde{\mu}) = \sum_{\bar{k}>k} D_{\bar{k},k}$$

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with

$$D_{\bar{k},k} = G_\nu^{1,k}(\nu_k, \mu_k; ...; \nu_{\bar{k}}, \mu_{\bar{k}}; \tilde{\nu}_{\bar{k}+1}, \tilde{\mu}_{\bar{k}+1}; ...; \tilde{\nu}_0, \tilde{\mu}_0) -$$
$$G_\nu^{1,k}(\nu_k, \mu_k; ...; \tilde{\nu}_{\bar{k}}, \tilde{\mu}_{\bar{k}}; \tilde{\nu}_{\bar{k}+1}, \tilde{\mu}_{\bar{k}+1}; ...; \tilde{\nu}_0, \tilde{\mu}_0)$$

and, by the inductive hypothesis and (**??**)

$$\sum_{\bar{k} \geq k} |D_{\bar{k},k}| \leq C_1 \bar{C} \varepsilon^2 \sum_{\bar{k} \geq k} \gamma^{\frac{1}{2}(k-\bar{k})} (\frac{\sigma_{\bar{k}}}{\gamma^{\bar{k}}})^2 \leq C_2 \bar{C} \varepsilon^2 (\frac{\sigma_k}{\gamma^k})^2 \tag{86}$$

so that the last sum in (83) is bounded by $\bar{C}\varepsilon(\frac{\sigma_h}{\gamma^h})^2$ with $\bar{C} = 4C$, and eq(84) is proved; by that equation it follows that for $h \geq h^*$ then $|\nu_h| \leq C\varepsilon$, so that it is possible to have that the flow of ν_h is bounded choosing a λ independent ν.

3.1.2 Proof that $\eta \equiv \eta(U)$

In a similar way we can prove that the critical index η is λ independent; remembering that

$$\frac{\sigma_{h-1}}{\sigma_h} = 1 + G_1(\lambda_h, \delta_h) + \frac{\sigma_h}{\gamma^h}\bar{G}_2$$

where to G_1 are contributing all the terms with relevant end-points and no non diagonal propagators. We introduce

$$\frac{\bar{\sigma}_{h-1}}{\bar{\sigma}_h} = 1 + G_1(\lambda_h, \delta_h)$$

such that $\bar{\sigma}_{h^*} = \sigma\gamma^{\eta h^*}$ with η of course σ independent. We write

$$\sigma_h = C_h \bar{\sigma}_h$$

By simple substitution it is in fact easy to see that

$$\frac{c_{h-1}}{c_h} = 1 + \frac{\bar{\sigma}_h}{\bar{\sigma}_{h-1}}[\frac{\sigma_h}{\gamma^h}\bar{G}_2]$$

so that

$$\frac{|c_{h-1}|}{|c_h|} \leq e^{C\varepsilon^2 \frac{\sigma_h}{\gamma^h}}$$

and

$$\frac{|c_{h-1}|}{|c_0|} \leq e^{C\varepsilon^2 \sum_{k=0}^{h} \frac{\sigma_k}{\gamma^k}} \leq 1 + O(\varepsilon^2)$$

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